Notes on Multiple Periodic Solutions for Second-order Discrete Hamiltonian System *

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Abstract

By a new orthogonal direct sum decomposition $E_M = Y \oplus Z$, which Z is related to $\Delta u_i (i = 1, 2, 3, ..., M)$, and a new functional $I(u)$, the method in [2] is improved to obtain new multiple periodic solutions with negativity hypothesis on F for a second-order discrete Hamiltonian system. Moreover, we exhibit an instructive example to make our result more clear, which hasn't been solved by the known results.

Keywords: Second-order Hamiltonian system; Periodic solutions; Critical points

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1 Introduction

The theory of nonlinear difference equation has been widely used to study discrete models appearing in many fields such as computer science, economic, neural networks, ecology, cybernetics, etc. In the past few years, the existence of periodic solutions of discrete systems is extensively investigated by critical point methods (see [1-6]).

In [1], Xue and Tang consider the second-order discrete Hamiltonian system

$$\Delta^2 u_{n-1} + \nabla_{u_n} F(n, u_n) = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $u_n = u(n)$, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, $F(t, x)$ is continuously differentiable in x for every $t \in \mathbb{Z}$ and T-periodic ($0 < T \in \mathbb{N}$) in t for all $x \in \mathbb{R}^N$, $\nabla_x F(t, x)$ is the gradient of $F(t, x)$ in x. By the minimax methods in the critical point theory, they establish the existence of at least one T-periodic solution.

In [2], Guo and Yu study the below difference equations

$$\Delta^2 u_{n-1} + f(n, u_n) = 0, \quad n \in \mathbb{Z},$$

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here $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $f(t+M,z)=f(t,z),\ (\forall\ (t,z)\in\mathbb{R}\times\mathbb{R})$, for some positive real number M. Employing the critical point theory, they develop a new method to gain the existence of periodic solutions. But, in [2], the authors presume that $\int_0^z f(t,\ s)ds > 0$.

It is noted that for the system (1.1), we discover that $F(n,u_n)\in\mathbb{R}$. However, so far as we known, there is no paper concerned with the case that $F\leqslant 0$ for multiple periodic solutions of second order discrete Hamiltonian system. For example, in [2] and [3], the authors presume that $\int_0^z f(t,s)ds > 0$ and $F\geqslant 0$ respectively. For more details in this direction, one consults to [1-6]. Besides, the gradient of F is related to u_{n-1}, u_n and u_{n+1}, so it is reasonable to substitute $\nabla_{u_n}(F(n,u_{n-1},u_n,u_{n+1}))$ for $\nabla_{u_n}F(n,u_n)$ in (1.1). Then in this paper, we are concerned with the following second order discrete Hamiltonian system:

$$\Delta^2 u_{n-1} + \nabla_{u_n}(F(n,u_{n-1},u_n,u_{n+1})) = 0, \quad \forall\, n\in\mathbb{Z}, \tag{1.2}$$

where $F:\mathbb{Z}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $F(t,\cdot,\cdot,\cdot)\in C^1(\mathbb{R}^3,\mathbb{R})$ for any fixed t, and there is a positive integer M, such that $F(n+M,x,y,z)=F(n,x,y,z), (\forall\ (x,y,z)\in\mathbb{R}^3,\ n\in\mathbb{N})$. By constructing a new orthogonal direct sum decomposition $E_M = Y\oplus Z$ which Z is related to $\Delta u_i (i=1,2,3,....,M)$, and a new functional $I(u)$, this paper is the first one to study multiple solutions with negativity hypothesis on F for (1.2).

Moreover, in Section 3, we exhibit a representative example to make our discussion on the system (1.2) more clear, which hasn't been solved by the known results.

2 Main results

Before establishing the existence of periodic solutions for system (1.2), we need to make some preparations.

- Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of all natural numbers, integers, and real numbers, respectively. For $a,\ b\in\mathbb{Z}$, define $\mathbb{Z}[a]=\{a,a+1,\ldots\}$, $\mathbb{Z}[a,b]=\{a,a+1,\ldots,b\}$ when $a\leqslant b$.
- Let S be the set of sequences, i.e. $S=\{u=\{u_n\}=(...,u_{-n},\ldots,u_0,\ldots,u_n,\ldots),\ u_n\in\mathbb{R},\ n\in\mathbb{Z}\}$. For any given positive integer M, E_M is defined as a subspace of S by

$$E_M=\{u=\{u_n\}\in S \mid u_{n+M}=u_n,\ n\in\mathbb{Z}\}. \tag{2.1}$$

- For $x,y\in S$, $a,b\in\mathbb{R}$, $ax+by$ is defined by

$$ax+by=\{ax_n+by_n\}_{n=-\infty}^{+\infty}.$$

  Then S is a vector space. Clearly, E_M is isomorphic to $\mathbb{R}^M$, E_M can be equipped with inner product

$$\langle x,y\rangle_{E_M}=\sum_{j=1}^{M}x_jy_j, \quad \forall\ x,y\in E_M.$$

  Then E_M with the inner product given above is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^M$. And the norms $\|\cdot\|$ and $\|\cdot\|_\beta$ induced by

$$\|x\|=(\sum_{j=1}^{M}x_j^2)^{\frac{1}{2}},\ \|x\|_\beta=(\sum_{j=1}^{M}|x_j|^\beta)^{\frac{1}{\beta}},\ \beta\in[1,\infty],$$

are equivalent, i.e., there exist constants $0 < C_1 \leqslant C_2$ such that

$$C_1\|x\| \leqslant \|x\|_\beta \leqslant C_2\|x\|, \quad \forall\, x \in E_M.$$

- $(\sum_{s=1}^{M}(\Delta u_s)^2)+2\Delta u_{n-1}\Delta u_n = (\Delta u_{n-1}+\Delta u_n)^2+\sum_{s=1}^{n-2}(\Delta u_s)^2+\sum_{s=n+1}^{M}(\Delta u_s)^2 = (\Delta u)^\top L(\Delta u)$, where $\Delta u = (\Delta u_1, \Delta u_2, \cdots, \Delta u_M)^\top$, L is a matrix. Then the eigenvalues of L are 0 and positive numbers, and we take $\gamma_{\min}$ representing the smallest positive eigenvalue of L. It is easy to get that $\eta = (0, 0, \ldots, 0, \Delta u_{n-1}, \Delta u_n, 0, \ldots, 0)^\top$ is a right eigenvector associated with the eigenvalue 0, if $0 \neq \Delta u_{n-1} = -\Delta u_n \in \mathbb{R}$.

- For a given matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{M\times M}, \tag{2.2}$$

  then $\xi = (v, v, \ldots, v)^\top$ is an eigenvector associated with the eigenvalue 0, if $0 \neq v \in \mathbb{R}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{M-1}$ be the other eigenvalues of A. Based upon the classical matrix theory and the results in [3], we have $\lambda_j > 0$ for all $j \in \mathbb{Z}[1, M-1]$. Moreover,

$$\lambda_{\min} = \min_{j\in\mathbb{Z}[1,M-1]}\lambda_j = 2(1-\cos\frac{2\pi}{M}) > 0, \qquad \lambda_{\max} = \max_{j\in\mathbb{Z}[1,M-1]}\lambda_j > 0. \tag{2.3}$$

- Denote by $Z = \{(u_1, u_2, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots, u_M) \in E_M \mid \Delta u_1 = \Delta u_2 = \ldots = \Delta u_{n-2} = 0,\ \Delta u_{n+1} = \Delta u_{n+3} = \ldots = \Delta u_M = 0, \Delta u_{n-1} = -\Delta u_n = -w, w \in \mathbb{R}\}$ and $Y = Z^\perp$, then $E_M = Y \oplus Z$ and $K = \{(v, v, \ldots, v)^\top \in E_M \mid v \in \mathbb{R}\}$ is contained in Z.

- Let X be a real Hilbert space, $I \in C^1(X, \mathbb{R})$, is said to satisfy Palais-Smale condition (P-S condition for short) if any sequence $\{u_n\} \subset X$ for which $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$, as $n \to \infty$, possesses a convergent subsequence in X.

Throughout this paper, we need the following assumptions: Let $M \geqslant 5$, and

(D_1) $F(t, x, y, z) \in C^1(\mathbb{R}^4, \mathbb{R})$ and for every $(t, x, y, z) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, F(t + M, x, y, z) = F(t, x, y, z)$;

(D_2) There exist constants $\delta > 0, b > 0$, such that:

$$F(n, u_{n-1}, u_n, u_{n+1}) \geqslant -b\lambda_{\min}(|u_{n-1}|^2 + |u_n|^2 + |u_{n+1}|^2),$$

for $n \in \mathbb{N}$ and $u_{n-1}^2 + u_n^2 + u_{n+1}^2 \leqslant \delta$, and $\lambda_{\min}$ is given in (2.3);

(D_3) There are constants $d_1 > 0, \beta > 2$ and $d_2 > 0$, such that

$$F(n, u_{n-1}, u_n, u_{n+1}) \leqslant \sum_{k=n-1}^{n+1} [-d_1|u_k|^\beta + d_2|u_k|^2].$$

Lemma 2.1 *Assume that assumption (D_3) holds, then the functional*

$$I(u) = \sum_{s=1}^{M} \frac{b+1}{\gamma_{\min}}(\Delta u_s)^2 + [\frac{2(b+1)}{\gamma_{\min}} + 1]\Delta u_{n-1}\Delta u_n + F(n, u_{n-1}, u_n, u_{n+1}) - G \tag{2.4}$$

is bounded from above on E_M, where

$$G = G(u_1, u_2, \dots, u_{n-2}, u_{n+2}, u_{n+3}, \dots, u_M) = b\lambda_{\min}[\sum_{s=1}^{n-2}|u_s|^\beta + \sum_{s=n+2}^{M}|u_s|^\beta]. \tag{2.5}$$

Proof. In fact,

$$\sum_{s=1}^{M}(\Delta u_s)^2 = \sum_{s=1}^{M}(u_{s+1} - u_s)^2 = \sum_{s=1}^{M}(2u_s^2 - 2u_s u_{s+1}). \tag{2.6}$$

So, by (D_3) and (2.5), for all $u \in E_M$,

$$\begin{aligned}
I(u) &= \sum_{s=1}^{M} \frac{b+1}{\gamma_{\min}}(\Delta u_s)^2 + [\frac{2(b+1)}{\gamma_{\min}} + 1]\Delta u_{n-1}\Delta u_n + F(n, u_{n-1}, u_n, u_{n+1}) - G \\
&\leqslant \sum_{s=1}^{M} [\frac{2(b+1)}{\gamma_{\min}} + 1](\Delta u_s)^2 + F(n, u_{n-1}, u_n, u_{n+1}) - G \\
&\leqslant \sum_{s=1}^{M} [\frac{2(b+1)}{\gamma_{\min}} + 1](\Delta u_s)^2 - \sum_{k=n-1}^{n+1}[d_1|u_k|^\beta - d_2|u_k|^2] - b\lambda_{\min}[\sum_{s=1}^{n-2}|u_s|^\beta + \sum_{s=n+2}^{M}|u_s|^\beta] \\
&\leqslant [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\sum_{s=1}^{M}(\Delta u_s)^2 - \min\{d_1,\, b\lambda_{\min}\}\sum_{s=1}^{M}|u_s|^\beta \\
&\leqslant [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\|\Delta u\|^2 - \min\{d_1,\, b\lambda_{\min}\}\|u\|_\beta^\beta \\
&\leqslant [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\sum_{s=1}^{M}(2u_s^2 - 2u_s u_{s+1}) - \min\{d_1,\, b\lambda_{\min}\}C_1^\beta\|u\|^\beta \\
&= [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]u^\top A u - \min\{d_1,\, b\lambda_{\min}\}C_1^\beta\|u\|^\beta \\
&\leqslant [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\lambda_{\max}\|u\|^2 - \min\{d_1,\, b\lambda_{\min}\}C_1^\beta\|u\|^\beta.
\end{aligned} \tag{2.7}$$

Since $\beta > 2$, from (2.7), there exists a constant $\widetilde{M} > 0$, such that for every $u \in E_M$, $I(u) \leqslant \widetilde{M}$. The proof is completed.

Lemma 2.2 *Assume that condition (D_3) is satisfied, then $I(u)$ satisfies P-S condition.*

Proof. Let $u^{(k)} \in E_M$, for all $k \in \mathbb{N}$, be such that $\{I(u^{(k)})\}$ is bounded. Then, by Lemma 2.1, there exists $M_1 > 0$, such that

$$-M_1 \leqslant I(u^{(k)}) \leqslant [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\lambda_{\max}\|u^{(k)}\|^2 - \min\{d_1,\, b\lambda_{\min}\}C_1^\beta\|u^{(k)}\|^\beta,$$

which implies

$$\min\{d_1,\, b\lambda_{\min}\}C_1^{\beta}\|u^{(k)}\|^{\beta}-[\frac{2(b+1)}{\gamma_{\min}}+1+d_2]\lambda_{\max}\|u^{(k)}\|^2\leqslant M_1.$$

For $\beta>2$, there is $M_2>0$ such that for every $k\in\mathbb{N}$, $\|u^{(k)}\|\leqslant M_2$.

Therefore, $\{u^{(k)}\}$ is bounded in E_M. Since E_M is finite dimensional, there exists a subsequence of $\{u^{(k)}\}$ (not labeled), which is convergent in E_M, so the P-S condition is verified.

Obviously, $I\in C^1(E_M,\mathbb{R})$. For any $u=\{u_n\}_{n\in\mathbb{Z}}\in E_M$, according to $u_0=u_M$, $u_1=u_{M+1}$, one computes that

$$\frac{\partial I}{\partial u_n}=\Delta^2 u_{n-1}+\nabla_{u_n}(F(n,u_{n-1},u_n,u_{n+1})),\quad \forall\, n\in\mathbb{Z}.$$

Therefore, the existence of critical points of I on E_M may implies the existence of periodic solutions of system (1.2). Initially, we give a lemma, which will serve us well later.

Lemma 2.3 (Linking Theorem) [4, Theorem 5.3]. *Let X be a real Hilbert space, $X=X_1\oplus X_2$, where X_1 is a finite-dimensional subspace of X. Assume that $I\in C^1(X,\mathbb{R})$ satisfies the P-S condition and*
(A_1) *there exist constants $\sigma>0$ and $\rho>0$, such that $I|_{\partial B_\rho\cap X_2}\geqslant\sigma$;*
(A_2) *there is an $e\in\partial B_1\cap X_2$ and a constant $R_1>\rho$, such that $I|_{\partial Q}\leqslant 0$,*
where $Q=(\bar{B}_{R_1}\cap X_1)\oplus\{re|\,0<r<R_1\}$, B_ρ denotes the open ball in X with radius ρ and centered at 0 and ∂B_ρ represents its boundary. Then, I possesses a critical value $c\geqslant\sigma$, here

$$c=\inf_{h\in\Gamma}\max_{u\in Q}I(h(u)),\Gamma=\{h\in C(\bar{Q},X)\mid h|_{\partial Q}=id\},$$

and id denotes the identity operator.

We are now in a position to state and prove our main result.

Theorem 2.1 *Let (D_1), (D_2) and (D_3) hold and $M\geqslant 5$. Then system (1.2) has at least two nontrivial M-periodic solutions.*

Proof. By Lemma 2.2, we obtain $\lim\limits_{\|u\|\to\infty}I(u)=-\infty$, so $-I$ is coercive. Let $c_0=\sup\limits_{u\in E_M}I(u)$. By the continuity of I on E_M, there exists $\bar{u}\in E_M$, such that $I(\bar{u})=c_0$, and $\bar{u}$ is a critical point of I. We claim that $c_0>0$. Indeed, for any $u=(u_1,u_2,\ldots,u_M)^\top$ with $u\in Y$,

$$\begin{aligned}
I(u) &= \frac{b+1}{\gamma_{\min}}\left[\sum_{s=1}^{M}(\Delta u_s)^2+2\Delta u_{n-1}\Delta u_n\right]+\Delta u_{n-1}\Delta u_n+F-G\\
&\geqslant \frac{b+1}{\gamma_{\min}}\left[\sum_{s=1}^{M}(\Delta u_s)^2+2\Delta u_{n-1}\Delta u_n\right]-\frac{1}{2}\sum_{s=1}^{M}(\Delta u_s)^2+F-G\\
&= \frac{b+1}{\gamma_{\min}}(\Delta u)^\top L(\Delta u)-\frac{1}{2}\sum_{s=1}^{M}(\Delta u_s)^2+F-G\\
&\geqslant (b+1)\|\Delta u\|^2-\frac{1}{2}\|\Delta u\|^2+F-G\\
&= (b+\frac{1}{2})\|\Delta u\|^2+F-G.
\end{aligned}$$

In view of (2.6), it yields

$$I(u) \geqslant (b+\frac{1}{2})\sum_{s=1}^{M}(2u_s^2 - 2u_s u_{s+1}) + F - G = (b+\frac{1}{2})u^\top Au + F - G,$$

where $u = (u_1, u_2, \ldots, u_M)^\top$, $\Delta u = (\Delta u_1, \Delta u_2, \ldots, \Delta u_M)^\top$, A is given by (2.2). Therefore, for any $u \in Y$, $\|u\| \leqslant \delta$, (δ is sufficiently small), by (D_2) and (2.5), one achieves

$$\begin{aligned} I(u) &\geqslant (b+\frac{1}{2})u^\top Au - b\lambda_{\min}(|u_{n-1}|^2 + |u_n|^2 + |u_{n+1}|^2) - b\lambda_{\min}\left[\sum_{s=1}^{n-2}|u_s|^\beta + \sum_{s=n+2}^{M}|u_s|^\beta\right] \\ &\geqslant (b+\frac{1}{2})\lambda_{\min}\|u\|^2 - b\lambda_{\min}(|u_{n-1}|^2 + |u_n|^2 + |u_{n+1}|^2) - b\lambda_{\min}\left[\sum_{s=1}^{n-2}|u_s|^2 + \sum_{s=n+2}^{M}|u_s|^2\right] \\ &= \frac{1}{2}\lambda_{\min}\|u\|^2. \end{aligned}$$

Take $\sigma = \frac{1}{2}\lambda_{\min}\delta^2$, then

$$I(u) \geqslant \sigma = \frac{1}{2}\lambda_{\min}\delta^2 > 0, \qquad \forall\, u \in Y \cap \partial B_\delta,$$

which hints that I satisfies the condition (A_1) in Lemma 2.3.

Note that when $u_1 = \ldots = u_M$, then $\Delta u_1 = \ldots = \Delta u_M = 0$. Combining (2.4) and (2.5), then

$$I(u) = F(n, u_{n-1}, u_n, u_{n+1}) - b\lambda_{\min}\left[\sum_{s=1}^{n-2}|u_s|^\beta + \sum_{s=n+2}^{M}|u_s|^\beta\right],$$

and it is clear that the maximum of I is 0, thus $I(u)$ does not acquire its maximum c_0. Therefore, the critical point associated with the critical value c_0 of I is a nonconstant M-periodic solutions of system (1.2). Of course, the critical point associated with the critical value c_0 of I is a nontrivial M-periodic solutions of system (1.2).

Finally, we show that (A_2) of the linking theorem holds. By Lemma 2.2, $I(u)$ meets P-S condition. Taking $e \in \partial B_1 \cap Y$, for any $z \in Z$, $r \in \mathbb{R}$, let $u = re + z$, from (2.7),

$$I(u) \leqslant [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\lambda_{\max}\|u\|^2 - \min\{d_1,\, b\lambda_{\min}\}C_1^\beta\|u\|^\beta.$$

Let

$$g_1(u) = [\frac{2(b+1)}{\gamma_{\min}} + 1 + d_2]\lambda_{\max}u^2 - \frac{\min\{d_1,\, b\lambda_{\min}\}C_1^\beta u^\beta}{2}, \quad g_2(u) = -\frac{\min\{d_1,\, b\lambda_{\min}\}C_1^\beta u^\beta}{2}.$$

Then

$$\lim_{u\to+\infty} g_1(u) = -\infty, \qquad \lim_{u\to\infty} g_2(u) = -\infty,$$

and g_1 and g_2 are bounded from above. Thus, there exists a big enough constant $R_2 > \delta$, such that $I(u) \leqslant 0$, for all $u \in \partial Q$, where

$$Q = (\bar{B}_{R_2} \cap Z) \oplus \{re|\, 0 < r < R_2\}.$$

With the aid of linking theorem, I exists a critical value $c \geqslant \sigma > 0$, where

$$c = \inf_{h\in\Gamma} \max_{u\in Q} I(h(u)), \qquad \Gamma = \{h \in C(\bar{Q}, E_M)|h|_{\partial Q} = id\}.$$

The rest of the proof is similar to that of [2, Theorem 1.1], but for the sake of completeness, we give the details.

Let $\tilde{u} \in E_M$ be a critical point associated to the critical value c of I, that is, $I(\tilde{u}) = c$. If $\tilde{u} \neq \bar{u}$, then the proof is complete. If $\tilde{u} = \bar{u}$, then $c_0 = I(\bar{u}) = I(\tilde{u}) = c$, that is

$$\sup_{u\in E_M} I(u) = \inf_{h\in\Gamma} \max_{u\in Q} I(h(u)).$$

Choose $h = id$, we have $\sup\limits_{u\in Q} I(u) = c_0$. Since the choice of $e \in \partial B_1 \cap Y$ is arbitrary, we can take $-e \in \partial B_1 \cap Y$. By a similar argument, there exists a constant $R_3 > \delta$ such that for any $u \in \partial Q_1$, $I(u) \leqslant 0$, where

$$Q_1 = (\bar{B}_{R_3} \cap Z) \oplus \{-re|\, 0 < r < R_3\}.$$

Again, by using the linking theorem, I possesses a critical value $c' \geqslant \sigma > 0$, where

$$c' = \inf_{h\in\Gamma_1} \max_{u\in Q_1} I(h(u)), \quad \Gamma_1 = \{h \in C(\bar{Q}_1, E_M)|\, h|_{\partial Q_1} = id\}.$$

If $c' \neq c_0$, then the proof is complete. If $c' = c_0$, then $\sup\limits_{u\in Q_1} I(u) = c_0$. Due to the fact that $I|_{\partial Q} \leqslant 0$, $I|_{\partial Q_1} \leqslant 0$, I attains its maximum at some points in the interior of the set Q and Q_1. Clearly, $Q \cap Q_1 = \emptyset$, and for any $u \in Z, I(u) \leqslant 0$. This shows that there must be a point $\hat{u} \in E_{qm}$, such that $\hat{u} \neq \tilde{u}$ and $I(\hat{u}) = c' = c_0$.

For all $\theta \in (0, 1)$, the above argument implies that whether $c = c_0$ or not, system (1.2) has at least two nontrivial M-periodic solutions.

Corollary 2.1 *Suppose that $F(n, u_{n-1}, u_n, u_{n+1})$ satisfies assumptions (D_1), (D_2) and the following assumption:*

(D_4) *there exist constants $d_1 > 0$, and $\beta > 2$, such that*

$$F(n, u_{n-1}, u_n, u_{n+1}) \leqslant -d_1(|u_{n-1}|^\beta + |u_n|^\beta + |u_{n+1}|^\beta).$$

Then, for a given positive integer $M \geqslant 5$, system (1.2) has at least two nontrivial M-periodic solutions.

Remark 2.1 *In Corollary* 2.1*, we mainly deal with the case that $F \leqslant 0$, which is different from the known results. This means that we obtain the existence of multiple periodic solutions for second-order discrete Hamiltonian system for a new and large range of the functional $F(n, u_{n-1}, u_n, u_{n+1})$.*

3 An example

We exhibit an representative example to make our discussion on the system (1.2) more clear.

Example 3.1. Take

$$F(n, u_{n-1}, u_n, u_{n+1}) = F(u_{n-1}, u_n, u_{n+1}) = -2b(1 - \cos\frac{2\pi}{M})[|u_{n-1}|^\beta + |u_n|^\beta + |u_{n+1}|^\beta],$$

where $\beta > 2$ is constant.

Proof. It is easy to see that $F \in C^1(\mathbb{R}^3, \mathbb{R})$, that (D_1), (D_2) and (D_3) are satisfied, thus system (1.2) possesses at least two nontrivial M-periodic solutions.

References

[1] Y.F. Xue, C.L. Tang, Existence of a periodic solution for subquadratic second-order discrete Hamiltonian system. Nonlinear Analysis: Theory, Methods & Applications, 2007, 67(7): 2072-2080.

[2] Z.M. Guo, J.S. Yu, Existence of periodic and subharmonic solutions for second-order superlinear differece equations. Science in China: Mathematics, 2003, 46(4): 506-515.

[3] Z. Zhou, J.S. Yu, Z.M. Guo, Periodic solutions of higher-dimensinal discrete systems. Proceedings of the Royal Society of Edinburgh-A-Mathematics. Edinburgh: The Society, 2004, 134(5): 1013.

[4] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations. American Mathematical Society Bookstore, 1986, 65.

[5] J.S. Yu, X.Q. Deng, Z.M. Guo, Periodic solutions of a discrete Hamiltonian system with a change of sign in the potential. Journal of Mathematical Analysis and Applications, 2006, 324(2): 1140-1151.

[6] P. Chen, H. Fang, Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations, Advances in Difference Equations, 2007.